CALCULATION OF NON-LINEAR FUNCTIONS FROM CERTAIN VARIABLES IN STUDYING THE STABILITY OF AN AUTOMATIC REGULATION SYSTEM

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[This article analyzes those conditions ~~are analyzed~~ necessary for the convergence of automatic regulation processes, following given (but not "small") initial deflections.

The method proposed earlier by the author is extended to the case where the differential equations of the process contain any number of non-linear functions, some of them in several variables.]

Generalizing somewhat the terminology introduced in the preceding work [1], we shall consider hereafter that the process of regulation undergoes a "decrement" in ~~the area~~ region L, if the equilibrium stabilized by the regulator and upset as a result of ~~the~~ an initial disturbance, characterized by any point ~~of the area~~ in region L, is reestablished some time later, after cessation of the disturbing effect ~~ceases~~. That is, the regulation process undergoes a decrement in L, if the equilibrium position stabilized by the regulator is asymptotically stable, and if all points belonging to the given ~~area~~ region of ~~the~~ initial deflections (L) also belong to the "~~area~~ region of stability for the most part" of this equilibrium position.

Conditions sufficient for a decrement of the system were determined in [1] for the case where the equations describing the regulation process contain one non-linear function in one variable; i. e. reducing to the form:

where a_{1j} and a_{ij} are constants ~~numbers~~ (some of which are zero), k is any number 1, 2, ..., n, and i is any number 2, 3, ..., n.

The generalization of the criteria found in [1] for the case where the system of equations describing the process contains several non-linear functions (each in one variable) is not difficult (a similar example was discussed in [1]).

The generalization of the same criteria for the case where the equations of the process contain any number of non-linear functions, including also functions in several variables, is the object of the present report.*

1. GENERALIZATION OF CRITERION I

Let us assume that the process of regulation is described by any system of equations:

Insert equation Page 21, [illegible] (1)

where F_i are assigned functions of all or some of the specified variables; $x_1, x_2, \ldots, x_n$ are increments of the generalized coordinates and velocities, reckoned from their equilibrium values in a regulated equilibrium position in such a way that the origin of the coordinates of the system's phase space (1) ($x_1 = x_2 = \ldots = 0$) corresponds to this

Footnote *In view of the fact that the present report is a continuation of report [1], the author has refrained from duplicating here a survey of the preceding works pertaining to the same problem of automatic regulation.

equilibrium and therefore $F_i(0,\dots,0) = 0$.

Let us simultaneously with equations (1) examine the linear equations

Insert eq. P 21 (2)

where a_{ij} are constants [redacted] (some of which are zero).

Concerning the system of equations (2) we shall assume only that it satisfies ~~the~~ Routh-Hurwitz' criteria ~~of Rous-Gurwitz~~. Otherwise, the coefficients a_{ij} may be selected arbitrarily, and this arbitrariness may be utilized in a practical application of the method given below.(*)

The considerations which permit us to generalize our criterion I are just generalizations of the considerations brought forth during the proving of this criterion [1].

Let us arbitrarily take the quadratic form

Insert eq P. 21 (3)

making ~~selecting~~ the ~~values of~~ constant coefficients $A_{x_i x_k}$ such ~~so as to ensure~~ that form (3) will be definitely negative.

Let us take a second quadratic form

Insert eq P. 21 (4)

and determine ~~determining~~ its coefficients $B_{x_i x_k}$ from the equations which we will obtain, ~~of~~ by equating the coefficients having similar terms in the relationship

Insert eq P. 21 (5)

NOTE TO TYPIST: NOTE THE DOT OVER X

where $\dot{x}_i$ is ~~are~~ taken in accordance with equations (2).

The phase curves of system (2) intersect any of the ellipsoids of the series $V = R$ (R is any non-negative number)

~~Footnote~~ *Thus the relationships (2) need not be equations of "small fluctuations" relative to system (1).

only from the outside inwards, since on the strength of (5) we have $\frac{dV}{dt} = U_A$, ~~but~~ and U_A is everywhere negative.

Let us examine now the system of equations

$$[illegible] \qquad (6)$$

where a_{ij} has the same values as in (5) and $\tilde{a}_{ij}$ are numbers determined below.

Retaining in (4) the above-found values of coefficients $P_{xi}x_k$, we shall find the derivative

$$[illegible] \qquad (7)$$

note dot

taking the values $\dot{\vec{x}}_i$ from system (6).

Then equation (7) will establish the derivative $\frac{dV}{dt}$ as a quadratic form relative to variables $x_1, x_2 \ldots, x_n$. Coefficients of this form (let us designate them $S_{xi}\,x_n$) will depend on $\tilde{a}_{ij}$, and the inequalities which have to be satisfied in order that the quadratic form (7) be definitely negative will bring conditions which in this case must be satisfied by $\tilde{a}_{ij}$:

$$[illegible] \text{ Equations } [illegible] \qquad (8)$$

where the quantities $\tilde{a}_{ij}$ and $\tilde{a}_{ij}$ are well defined(*) by the coefficients a_{ij} and $A_{xi}\,x_k$, which may be selected arbitrarily, limited only by the above indicated conditions.

System (6), together with inequalities (8), defines the set of linear equations, whose phase curves intersect any of the ellipsoids of the system $V = R$ only from the outside inwards.

Footnote (*) It is essential only that $\tilde{a}_{ij} = 0$ should satisfy inequalities (8).

let us now substitute in the derivative

values x_i determined by the examined system (1). It is not difficult to notice that $\frac{dV}{dt}$ continues to remain negative at any point in the phase space, if the quantities a_{ij}, satisfying the inequality can be so selected (8), that at this point the value of any of the nonlinear functions F_i $(x_1, x_2, \ldots, x_n)$ coincide with the values of the corresponding linear function

We shall select from among the ellipsoids of the series $V = R$ any ellipsoid $V = R_1$, and name the series of points inside this ~~area~~ region as "~~area~~ region R."

If all $F_i(x_1, x_2, \ldots, x_n)$ of the investigated nonlinear system were such that, for any point $x_{1o}, x_{2o}, \ldots, x_{no}$ of ~~area~~ region R_1 of the phase space, it is possible to find among the quantities a_{ij} satisfying inequalities (8) such values (perhaps for each of its points) that

(9)

then the derivative $\frac{dV}{dt}$ throughout the ~~area~~ region R_1 expressed by system (1) is definitely negative and the curves of system (1) intersect the set of ellipsoids $V = R$ only from outside inwards, converging towards the origin of the coordinates and, consequently, system (1) in this case, undergoes a decrement in ~~area~~ region R_1.

Generalizing somewhat the considerations brought forth, we can put the result obtained ~~can be given~~ in the form of the following criterion:

Generalized Criterion I

If it is possible for a linear system of differential equations with constant coefficients (2) to construct a definitely positive "Lyapunov"* function V whose derivative $\frac{dV}{dt}$ expressed by system (2), will be a definitely negative function for any values of a_{ij} satisfying the inequalities

then, in order for the non-linear system (1) to undergo a decrement in any region $V = R_1$ of the phase space, it will be sufficient if for every point of this region it is possible to select such values of a_{ij} satisfying the indicated inequalities, at which the value of $\sum_{j=1}^{j=n} a_{ij} x_j$ coincides with the value of $F_i(x_1, x_2, \ldots, x_n)$.

2. NUMERICAL EXAMPLE

Formulation of Example

Taking into account the non-linearity of the characteristics of the engine, and disregarding the non-linearity of the characteristics of the regulator, the object of this example is to determine those conditions, for the below-described installation consisting of a high-speed Diesel engine with direct speed regulation, which are sufficient to cause convergence of the regulation process after any initial deviation produced by a momentary change in the engine speed of not more than 100 rpm and a displacement of the regulator coupling of not more than 3mm*2. Initial speed of the coupling is equal to zero.

Footnotes: * The term "Lyapunov function" is here understood to be in the restricted meaning indicated in [1] and not in the broader sense which is usually understood in this connection.

*2 Similar initial deflections can be caused for instance by short-duration load changes.

Description of the installation and the equation for the process.

The high-speed Diesel engine is utilized for fixed operation at $n = 1200$ r.p.m. A resistance (= load torque =) moment is applied to the engine which in effect does not ~~change~~ vary during momentary changes in ~~the~~ speed ~~of the~~ engine, and is equal to 16 kg · m of torque.

NOTE: subscript "g"

Figure 1 and 2 show two variants of engine characteristics ~~are shown~~. Each curve determines the ~~change of the existing~~ variation in torque-moment ΔM_g (computed ~~from~~ equilibrium $M_g = 16$ kg · m) ~~with a change~~ in the rpm range ~~from~~ -300 to + 300 rpm (computed from equilibrium $n_e = 1200$ rpm) and with an unchanging (but different) position of the coupling of the regulater, which is disconnected from the engine.

subscript "g"

The moment of inertia I_g of all the engine's retating and forward moving masses, brought out in its flywheel, equals 0.24 kg · m² · sec².

The equation of motion of the engine has the form

or [illegible]

[illegible] 25 →

where F(x,y) is a function, defined by ~~a series~~ the family of curves shown in Figure 3 (first variant of the engine) and in Figure 4 (second variant of the engine), obtained from Figures 1 and 2 by changing the scale of the ordinate axis 39.9 times.

A centrifugal regulater of construction "NATI" (Figure 5) is used in the capacity of a direct-action regulater. The parameters of the regulater are such(*) that the equation of the regulater has the form

[illegible] equation 9B [illegible]

In these equations and also in Figures 1, 2, 3, ~~and~~ and 4: $x = \Delta n$ are the

Footnote: (*) All data on regulator "NATI" were submitted to the author by Prof. G. G. Kalish.

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(increment - decrement) equilibrium center
deviations in the engine's rotation computed from 1200 rpm ($x > 0$ if the rotations increase); y is the displacement of the regulator coupling, computed from the equilibrium position corresponding to engine operation at 1200 rpm.

Thus, the process of regulation is described by the equations

(10)

In the phase space with coordinates i.e. x, y, ξ, the region of initial deviations is given by the rectangle $-100 < x < + 100$, $-3 < y < + 3$, lying in the plane $\xi = 0$.

DETERMINATION OF CONDITIONS FOR A DECREMENT OF THE SYSTEM.

We shall take as a linear approximation of equations (10)

(11)

placing $a = 0$ and $b = 50$ and taking into consideration the course of the curves in Figures 3 and 4.

This system is similar to system (21) of report [1], for which in [1] the set of ellipsoids of interest to us was constructed to intersect the curves of the linear system only from the outside inwards.

In our case, $Na = 0$, $b = 50$, $h = 100$, $c = 6872$ and $k = 104.6$. Repeating the computations carried out in [1], but taking into account these values for the coefficients and setting U_A in the form $U_A = -A(x^2 + y^2 + \xi^2)$ for the equation of the set of ellipsoids

(12)

we easily / ~~it is not difficult to~~ determine the following values for the coefficients:

(13)

Let us now examine equations

(14)

8.

and determine the derivative

[illegible]

using the values $\dot{x}$, $\dot{y}$ and $\dot{\xi}$ from system (14).

For this derivative the quadratic form U_s =

[illegible] (15)

is employed with coefficients equal to

[illegible] (16)

The quadratic form (15) is definitely negative if

and [illegible]

On the strenght of (16), these inequalities are ~~fulfilled~~ satisfied if

and [illegible]

It is easily demonstrated that any Na and b satisfy these inequalities, if

Insert eq (17) (17)

In order to determine whether the conditions for the above-proven criteria are fulfilled, we must new compare, passing ever to equations (10), the non-linear function F = 39.9F (x,y) given by the series of curves in Fig 3 (or in Fig (4)) with the linear function

Insert 17A, Page 28

It is necessary to establish ~~at~~ for which values of x and y it is possible to select Na and b, satisfying the inequalities (17), in such a way that

$$39.9\, F(x,y) = -\tilde{N}ax - (50 + \tilde{b})y.$$

With this purpose in mind, a ~~series~~ family of straight lines

$$F = a_1^{*}x - 50.36y \text{ and } F = a_2^{*}x - 46y$$

were drawn, (in Figure 6 for first variant of the engine and in Figure 7 for the second variant) where

and y was given all those values, in turn, ~~at~~ for which the curves illustrated in Figures 2 and 3 were drawn (these values are y = -3, -3, -1, -, 1, 2, 3). ~~and 3)~~

Thus, each of the indicated values for y corresponds, on the one hand, to a specific curve ~~on~~ in Figure 2 (or on Figure 3), and, on the other hand, to a region limited by 4 segments of straight lines*, shown in Figure 6 (or in Figure 7).

One of these regions, (corresponding to y = -3) is cross-hatched in Figures 6 and 7**.

Conditions for our criterion are fulfilled in this case if all the curves assigned to function F (x, y) do not emerge beyond the limits of the regions constructed for them in this manner.

Footnotes: *For case y = 0 with two straight lines.

**The number of these regions to be constructed is the same as the number of curves ~~which were~~ assigned to the non-linear function F (x,y) being considered.

***Within limits, determined by the range of fixed characteristics.

SOLUTION OF EXAMPLE

From Figure 6 it follows~~,~~ that, in regulating an engine whose characteristics are shown in Fig. 2, the conditions for our criterion are fulfilled and the process of regulation converges after any initial deflection, whatever they might be ***. (NOTE TO TYPIST: See preceding page for footnote).

In regulating an engine whose characteristics are shown in Figure 3, the conditions of our criterion, in accordance with Figure 7, are fulfilled only for ~~at~~ $x > -200$.

Let us separate from among the ellipsoids (12), with values B determined by equations (13), that ~~the~~ ellipsoid, related to ~~the~~ plane $x = -200 = \text{const.}$, and let us cut this ellipsoid by the plane $\xi = 0$.

In the cross-section there appears an ellipse

$$0.6775x^2 + 35.37y^2 - 2.018xy = 27795.8$$

The region of initial deflections, assigned by the conditions of the example

$$(18)$$

lies entirely within this ellipse, and, accordingly, the tested regulation process, described by equations (10), converges after any initial deflection satisfying the given conditions (18).

STATING A NEW PROBLEM

The region ~~where~~ through which the non-linear characteristics can pass ~~through~~ arbitrarily without violating the necessary conditions for a decrement of the system becomes more extensive, the stronger the inequalities in ~~divergence in~~ the values of ~~quantities~~ a_{ij}* and a_{ij}** entering ~~into inequalities~~ (8). It is natural, therefore, to explain the limiting values for these quantities. At the same time, it ~~there~~ is no longer necessary to ~~any basis for~~ connect~~ing~~ them with the proven criterion, and to require that, ~~for~~ system (6) ~~, it should be~~

~~possible to~~ can use the same "Lyapunov" function for any a_{ij} satisfying the inequalities (8). It is essential only that system (1) should undergo a decrement in the entire space, if ~~it is possible to select~~ a_{ij} can be selected for any point in such a way that the values ~~of~~ in the right ~~parts~~ side of equations (1) and (6) ~~would~~ coincide.

Let system (2) satisfy ~~the~~ Routh-Hurwitz' criterion ~~of Raus-Gurvits at~~ for any a_{ij}, if only

[illegible]

and does not satisfy it, if $a_{ij} < b^{*}_{ij} - \varepsilon$ or $a_{ij} > b^{**}_{ij} + \varepsilon$ however small the positive number ε may be.

Is it at all possible to widen the band between a_{ij}^{*} and a_{ij}^{**} up to the limits of ~~Raus-Gurvits~~ Routh-Hurwitz; i. e., up to the values b_{ij}^{*} and b_{ij}^{**}?

We shall limit ourselves here ~~only with~~ merely to the statement of this interesting problem, whose solution would not only ~~permit a~~ considerably simplify ~~simplification of~~ the above-stated method, but ~~would~~ also ~~permit the~~ develop ~~development~~, fundamentally, the basis of linear methods, which the author considers as the most important result of the present work.

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1. M. A. Ayzerman, Convergence of processes of automatic regulation after large initial deflections, Automatics and Telemechanics, 2 -3, (1946).

Fig. 1 Assigned characteristics of a Diesel-motor (first variant)

Fig. 2 Assigned characteristics of a Diesel-motor (second variant)

Fig. 3 Value of the function 39.9 F (x,y) for a Diesel engine (first variant)

Fig. 4 Value of the function 39.9 F (x, y) for a Diesel engine (second variant)

Fig. 5 Principle diagram of Diesel engine regulator

Lever for changing the fixed conditions

Pump rod

Engine transmission

Fig. 6 Comparison of Diesel engine characteristics (first variant) with ~~areas~~ regions through which these characteristics must pass in order that the process ~~of~~ regulation will converge after any initial deflection.

Fig. 7 Comparison of Diesel engine characteristics (second variant) with ~~areas~~ regions through which these characteristics must pass in order that the process ~~of~~ regulation will converge after any initial deflection.

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